Monday, March 20, 9:00-9:40

# A SEMIGROUP APPROACH TO NONLINEAR LÉVY PROCESSES

ROBERT DENK

A Lévy process $(X_t^{(\lambda)})_{t\geq 0}$ in $\mathbb{R}^d$ with Lévy triplet $\lambda = (b, Q, \nu)$ generates a $C_0$-semigroup in $\mathrm{BUC}(\mathbb{R}^d)$ which is defined by

$$(S_\lambda(t)f)(x) = \mathbb{E}\big[f(x + X_t^{(\lambda)})\big]. \tag{1}$$

Moreover, the function $t \mapsto u(t) := S_\lambda(t)f$ yields a solution of the Cauchy problem $\partial_t u - A_\lambda u = 0,\ u(0) = f$, where $A_\lambda$ denotes the generator of $(S_\lambda(t))_{t\geq 0}$.

In this talk, we consider the situation of model uncertainty, where it is only assumed that the Lévy triplet of the process belongs to some set $\Lambda$ of Lévy triplets. For this, one considers $(J(t)f)(x) := \sup_{\lambda\in\Lambda} \mathbb{E}\big[f(x + X_t^{(\lambda)})\big]$ and the corresponding fully nonlinear PDE

$$\partial_t u - \sup_{\lambda\in\Lambda} A_\lambda u = 0,\ u(0) = f. \tag{2}$$

To generalize (1) to the nonlinear situation, we first establish a one-to-one relation between Lévy processes under convex expectations $\mathcal{E}$ and nonlinear Markovian convolution semigroups. In a second step, we refine the definition of $(J(t))_{t\geq 0}$ to obtain a nonlinear semigroup $(\mathscr{S}(t))_{t\geq 0}$, following an approach due to Nisio. In this way, we obtain that for any bounded family of Lévy triplets, there exists a nonlinear semigroup $(\mathscr{S}(t))_{t\geq 0}$ on $\mathrm{BUC}(\mathbb{R}^d)$, a convex expectation space $(\Omega, \mathscr{F}, \mathcal{E})$ and an $\mathcal{E}$-Lévy process $(X_t)_{t\geq 0}$ such that

$$u(t,x) := (\mathscr{S}(t)f)(x) = \mathcal{E}\big[f(x + X_t)\big]$$

defines a viscosity solution of the fully nonlinear PDE (2).

This talk is based on the joint paper [1] with Michael Kupper (Konstanz) and Max Nendel (Bielefeld).

## References

[1] R. Denk, M. Kupper, and M. Nendel. A semigroup approach to nonlinear Lévy processes. *Stochastic Process. Appl.*, 130(3):1616–1642, 2020.

Monday, March 20, 9:45-10:25

## NON-LOCAL PERIMETER AND CURVATURE

TOMASZ GRZYWNY

We introduce a notion of non-local perimeter which is defined through an arbitrary positive Borel measure on $\mathbb{R}^d$ which integrates the function $1 \wedge |x|$. Such definition of non-local perimeter encompasses a wide range of perimeters which have been already studied in the literature, including fractional perimeters and anisotropic fractional perimeters. Furthermore, for a symmetric absolutely continuous measure $\nu$ that a density is comparable with a radial non-increasing function, we define a non-local curvature. These objects can be viewed as the point-wise acting of the generator of transition semigroup of a Lévy process. The main part of the talk will be devoted to the study of the asymptotic behaviour of non-local perimeters and curvatures. As direct applications we recover well-known convergence results for fractional and anisotropic fractional perimeters and curvatures.

The talk will be based on the joint project with Wojciech Cygan (TU Dresden) and Julia Lenczewska (WUST).

Monday, March 20, 11:00-11:40

# NONLOCAL CAHN-HILLIARD TYPE EQUATIONS AND THEIR LOCAL LIMITS

HELMUT ABELS

Cahn-Hilliard type equations are used to describe phase separations in various physical systems. Besides the classical "local" Cahn-Hilliard equation, which leads a fourth order parabolic equation with non-monotone nonlinearity, there is a "nonlocal" conter-part, which involves a non-convolution type operator instead of a Laplacian in the equation of the so-called chemical potential. It has the advantage that it can be derived as a limit of micro-scopic models. We discuss known analytic results for both types of Cahn-Hilliard equations and corresponding Navier-Stokes/Cahn-Hilliard systems, which describe diffuse interface models for the two-phase flow of viscous incompressible fluids. Moreover, we present results on convergence of the nonlocal systems to their local counterparts.

## References

[1] H. Abels, S. Bosia, and M. Grasselli, Cahn-Hilliard equation with nonlocal singular free energies, *Ann. Mat. Pura Appl.* (4), **194** (2015), 1071–1106.
[2] H. Abels and Y. Terasawa, Weak solutions for a diffuse interface model for two-phase flows of incompressible fluids with different densities and nonlocal free energies, *Math. Methods Appl. Sci.*, **43** (2020), 3200–3219.
[3] E. Davoli, L. Scarpa, and L. Trussardi, Local asymptotics for nonlocal convective Cahn-Hilliard equations with $W^{1,1}$ kernel and singular potential, *J. Differential Equations*, **289** (2021), 35–58.
[4] E. Davoli, L. Scarpa, and L. Trussardi, Nonlocal-to-local convergence of Cahn-Hilliard equations: Neumann boundary conditions and viscosity terms, Arch. Ration. Mech. Anal., **239** (2021), 117–149.
[5] S. Frigeri, On a nonlocal Cahn-Hilliard/Navier-Stokes system with degenerate mobility and singular potential for incompressible fluids with different densities, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **38** (2021), 647–687.
[6] G. Giacomin and J. L. Lebowitz. Phase segregation dynamics in particle systems with long range interactions. I. Macroscopic limits. *J. Statist. Phys.*, 87:37–61, 1997.
[7] Z. Liang, On the existence of weak solutions to non-local Cahn-Hilliard/Navier-Stokes equations and its local asymptotics, *Commun. Math. Sci.* **18** (2020), no. 8, 2121–2147.
[8] A. C. Ponce, A new approach to Sobolev spaces and connections to Γ-convergence, *Calc. Var. Partial Differential Equations*, **19** (2004), 229–255.

Monday, March 20, 11:45-12:25

# SEMI-MARKOV PROCESSES AND THE NON-LOCAL HEAT EQUATION ON A TIME DEPENDENT DOMAIN

BRUNO TOALDO

We first introduce the theory of semi-Markov processes. In particular we discuss a general theory to construct semi-Markov processes by means of a suitable random time-change ([8, 10]). A prototype example of this theory are processes $X = (X(t))_{t\geq 0}$ for $X(t) := M(L(t))$ where $M = (M(s))_{s\geq 0}$ is Feller process and $L = (L(t))_{t\geq 0}$ is the inverse (hitting-time) process of a subordinator $\sigma = (\sigma(s))_{s\geq 0}$, i.e.,

$$L(t) := \inf \{s > 0 : \sigma(s) > t\} .$$

Then we describe, in general, the interplay of these semi-Markov processes with non-local equations (see, e.g., [2, 3, 4, 5, 6, 7, 9, 10]). We will focus, in particular, on a non-local (in time) heat equation on a time-increasing parabolic set whose boundary is determined by a suitable curve ([1]). We provide a notion of solution for this equation and we study well-posedness under Dirichlet conditions outside the domain. A maximum principle is proved and used to derive uniqueness and continuity with respect to the initial datum of the solutions of the Dirichlet problem. Existence is proved by showing a stochastic representation based on the delayed Brownian motion (a suitable semi-Markov process) killed on the boundary.

## References


[1] G. Ascione, P. Patie and B. Toaldo. Non-local heat equation with moving boundary and curve-crossing of delayed Brownian motion. *Submitted*, available at arXiv:2203.09850, 2022.

[2] B. Baeumer and M.M. Meerschaert. Stochastic solutions for fractional Cauchy problems. *Fractional Calculus and Applied Analysis* 4: 481–500, 2001.

[3] Z.-Q. Chen. Time fractional equations and probabilistic representation. *Chaos, Solitons and Fractals*, 102: 168 – 174, 2017.

[4] M. E. Hernández-Hernández, V.N. Kolokoltsov and L. Toniazzi. Generalised Fractional Evolution Equations of Caputo Type. *Chaos, Solitons & Fractals,* 102: 184 – 196, 2017.

[5] A.N. Kochubei. General fractional calculus, evolution equations and renewal processes. *Integral Equations and Operator Theory*, 71: 583 – 600, 2011.

[6] V.N. Kolokoltsov. Generalized Continuous-Time Random Walks, subordination by hitting times, and fractional dynamics. *Theory Probab. Appl.* 53:, 594–609, 2009.

[7] M. Magdziarz and R.L. Schilling. Asymptotic properties of Brownian motion delayed by inverse subordinators. *Proc. Amer. Math. Soc.*, 143: 4485 – 4501, 2015.

[8] M.M. Meerschaert and P. Straka. Semi-Markov approach to continuous time random walk limit processes. *The Annals of Probability*, 42(4) : 1699 – 1723, 2014.

[9] P. Patie and A. Srapionyan. Self-similar Cauchy problems and generalized Mittag-Leffler functions. *Frac. Calc. Appl. Anal.*, 24(2): 447 - 482, 2021.

[10] M. Savov and B. Toaldo. Semi-Markov processes, integro-differential equations and anomalous diffusion-aggregation. *Annales de l'Institut Henri Poincaré (B) Probability and Statistics*, 56(4): 2640 – 2671, 2020.

Monday, March 20, 12:30-13:10

# MAPPING PROPERTIES OF FRACTIONAL INTEGRALS AND DERIVATIVES

DAVID BERGER (CAILING LI AND RENÉ SCHILLING.)

In this talk we will discuss mapping properties of Riemann-Liouville integrals and derivatives by using interpolation theory.
The Riemann-Liouville integral $I_0^\alpha$ is defined by

$$I_0^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_0^x f(t)(x-t)^{\alpha-1} dt$$

and the Riemann-Liouville derivative is given by

$$D_0^\alpha f(x) := \frac{d}{dx} I^{\alpha-0} f(x).$$

We first give an introduction to interpolation theory and fractional derivatives. We speak about known properties of the Riemann-Liouville integrals. From there we discuss certain Lipschitz type spaces introduced by D.E. Edmunds and D.D. Haroske and draw a connection to the usual Besov and Sobolev spaces. We will later use these spaces to derive mapping properties of the Riemann-Liouville integrals in certain cases. At last we shortly talk about extensions to more general operators.

Monday, March 20, 15:45-16:25

# SEMICONVEXITY AND THE NONLOCAL OBSTACLE PROBLEM

MARVIN WEIDNER (JOINT WORK WITH XAVIER ROS-OTON, DAMIÀ TORRES-LATORRE)

The Bernstein technique is a classical tool to establish derivative estimates for solutions to a large class of elliptic and parabolic equations. The main idea of the method is to apply the maximum principle to certain suitable auxiliary functions that depend on the derivatives of a solution to the PDE under consideration.
In the first part of the talk, we explain how the Bernstein technique can be extended to integro-differential equations driven by nonlocal operators of the form

$$Lu(x) = \text{p.v.} \int_{\mathbb{R}^n} (u(x) - u(y))K(x-y)\mathrm{d}y,$$

where $K : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty]$ is a kernel satisfying

$$\lambda(1-s)|y|^{-n-2s} \le K(y) \le \Lambda(1-s)|y|^{-n-2s} \ \forall y \in \mathbb{R}^n,$$

where $s \in (0,1)$ and $0 < \lambda \le \Lambda$.

In the second part of the talk, we explain how the Bernstein technique can be used to establish semiconvexity estimates for solutions to the nonlocal obstacle problem

$$\min(Lu, u-\phi) = 0 \quad \text{in } \Omega, \tag{1}$$

where $\phi$ is a smooth obstacle and $\Omega \subset \mathbb{R}^n$ is a bounded domain. As an application of this result, we establish the optimal $C^{1+s}$-regularity of the solution $u$ and the regularity of the free boundary $\partial\{u = \phi\}$ near regular points.

## References

[1] Xavier Ros-Oton, Damià Torres-Latorre, Marvin Weidner, *Semiconvexity estimates for nonlinear integro-differential equations.* work in progress, 2023.

Tuesday, March 21, 9:00-9:40

# THE NONLOCAL NEUMANN PROBLEM, TRACES SPACES AND REFLECTING JUMP PROCESSES

MORITZ KASSMANN

We derive a probabilistic representation of solutions to nonlocal parabolic equations with Neumann-type conditions on the complement of a given bounded domain. We make use of recent progress on the Hilbert space setting of nonlocal Neumann problems. We explain the new results on trace spaces by Grube/Hensiek and Grube/Kassmann. We introduce a Markov jump process that is well suited to model reflections from the complement of a domain back into the domain. Our approach is robust in the sense that it allows for an approximation of reflecting Brownian Motion. The talk is based on a joint work with Soobin Cho.

Tuesday, March 21, 9:45-10:25

# APPROXIMATIONS OF LÉVY PROCESSES ON BOUNDED DOMAINS

BORIS BAEUMER, MIHÁLY KOVÁCS, MATT PARRY, AND LORENZO TONIAZZI

We identify Dirichlet and no-flux boundary conditions for the governing equations for one-sided Lévy processes on an interval which are modified to either be killed, re-inserted a the boundary, or time-changed to remain in the interval. This is achieved by using finite state approximations stemming from Lubich's convolution quadrature, a Trotter-Kato convergence argument of the corresponding generators and semigroups, and a use of the Continuous Mapping Theorem with regards to the Skorokhod topology. We'll conjecture that these approximations extend to all Lévy processes (not just the one-sided ones) modified in the same way.

In the stable case with Dirichlet boundary conditions the first eigenmode behaves like $x^{\alpha-1}$ at the boundary [3, 1], destroying first order convergence of commonly used approximation schemes for the $\alpha$-derivative operator like the Grunwald approximation. This led to the discovery of a new scheme that is positivity preserving and of order $\alpha$, involving the polylogarithm and Riemann Zeta function. [2]

## References

[1] Boris Baeumer, Mihály Kovács, Mark M. Meerschaert, and Harish Sankaranarayanan. Boundary conditions for fractional diffusion. *Journal of Computational and Applied Mathematics*, 336:408 – 424, 2018.

[2] Boris Baeumer, Mihály Kovács, and Matthew Parry. A higher-order resolvent positive finite difference approximation for fractional derivatives on bounded domains. *Fractional Calculus and Applied Analysis*, 25:299–319, 2022.

[3] Boris Baeumer, Mihály Kovács, and Harish Sankaranarayanan. Fractional partial differential equations with boundary conditions. *Journal of Differential Equations*, 264(2):1377–1410, 2017.

Tuesday, March 21, 11:00-11:40

# CONCATENATION OF DISHONEST FELLER PROCESSES, EXIT LAWS, AND LIMIT THEOREMS ON GRAPHS

ADAM BOBROWSKI

We provide a rather explicit formula for the resolvent of a concatenation of $N$ processes in terms of their exit laws and certain probability measures characterizing the way the processes are concatenated. Here are the details: Suppose we are given $N$ Feller non-honest processes $X_i, ..., X_N$ in $N$ separate (i.e., disjoint) compact spaces $S_1, \dots, S_N$. Suppose also that the $i$th process has a regular Feller boundary with $k(i)$ exits; the $j$th exit law of the $i$th process is denoted $\ell_\lambda^{i,j}, \lambda > 0$ (this is the Laplace transform of the time needed to exit from $S_i$ through the $j$th gate). By *disjoint union* of these processes we understand the process with state-space $S_{\mathrm{u}} := \dot{\bigcup}_{i\in\mathcal{N}} S_i$ which after starting at an $x \in S_i$ behaves like $X_i$, and is left undefined when $X_i$ is undefined; the related Feller resolvent in $C(S_{\mathrm{u}})$ is denoted $R_\lambda^{\mathrm{du}}, \lambda > 0$. Our main result says that then the formula

$$R_\lambda^{\mathrm{co}} f = R_\lambda^{\mathrm{du}} f + \overset{\cdot}{\sum_{i\in\mathcal{N}}} \sum_{j=1}^{k(i)} \left( \int_S R_\lambda^{\mathrm{co}} f \,\mathrm{d}\mathfrak{p}_{i,j} \right) \ell_\lambda^{i,j}, \qquad \lambda > 0, f \in C(S_{\mathrm{u}}), \tag{1}$$

where $\mathfrak{p}_{i,j}$ are given probability measures in $S_{\mathrm{u}}$, defines a Feller resolvent in $C(S_{\mathrm{u}})$. The related process, termed *concatenation* of $X_1, \dots, X_N$, is governed by the following rules.

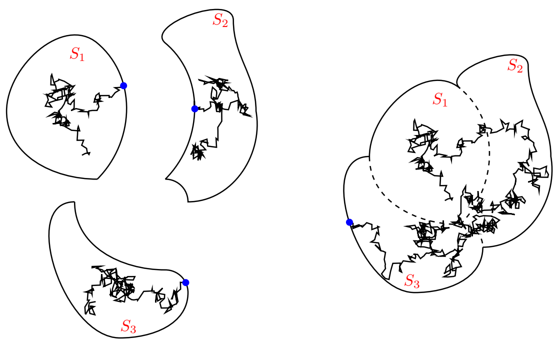
(i) Conditional on starting at an $x \in S_i$, the process is identical to $X_i$ up to the random time when $X_i$ is no longer defined.

(ii) If and when the process $X_i$ is no longer defined, the concatenation starts afresh at a random point $y \in S_{\mathrm{u}}$, the distribution of its position at this moment depending on the gate through which $S_i$ was exited. More specifically, this position is described by a Borel probability measure $\mathfrak{p}_{i,j}$ on $S_{\mathrm{u}}$. In most applications, $\mathfrak{p}_{i,j}$ is supported in $S_{\mathrm{u}} \setminus S_i$, but this assumption is not needed in the proof.

(iii) If $y$ belongs to $S_j$, the concatenation from that time on is identical to $X_j$ up to the random time when $X_j$ is no longer defined, and so on.

Relation (1) is a reflection of strong Markov property of the concatenation. Technically, is seems to define the uknown $R_\lambda^{\mathrm{co}}$ via unknown integrals $\int_S R_\lambda^{\mathrm{co}} f \,\mathrm{d}\mathfrak{p}_{i,j}$. The latter, however, can be beforehand calculated from a consistency relation which comes down to a linear system of equations with $\int_S R_\lambda^{\mathrm{du}} f \,\mathrm{d}\mathfrak{p}_{k,l}$ and $\int_S \ell_\lambda^{i,j} \mathfrak{p}_{k,l}$ as given data.

As an application, we obtain an averaging principle saying that by concatenating asymptotically splittable processes one can approximate Markov chains.




Processes before (on the left) and after concatenation (on the right).

Tuesday, March 21, 11:45-12:25

# THE FRACTIONAL LAPLACIAN WITH REFLECTIONS

MARKUS KUNZE

Denote the $\alpha$-stable Lévy Process by $Y = (Y_t)_{t\geq 0}$ and let $D$ be an open subset of $\mathbb{R}^d$ with Lipschitz boundary. Consider the stochastic process $X = (X_t)_{t\geq 0}$ which is constructed from $Y$ as follows. As long as $Y$ stays within $D$, the process $X$ equals $Y$. However at the time $\tau_D$ of first exit from $D$, instead of moving to $z = Y_{\tau_D}$, the process $X$ is restarted (immediately and without even visiting $D^c$) at a point $y \in D$ which is chosen (randomly) according to a probability measure $\mu(z, dy)$.

In this talk, we will present some (analytical) results concerning the transition semigroup $(K_t)_{t\geq 0}$ of the process $X$. In particular, we will give a complete characterization of the generator of the semigroup.

This talk is based on joint work with Krzysztof Bogdan.

## References

[1] Bogdan, K. and Kunze, M., *The fractional Laplacian with reflections*, preprint (2022). Available at https://arxiv.org/abs/2211.05511.

Tuesday, March 21, 12:30-13:10

# PARAMTRIX CONSTRUCTION FOR A LÉVY-TYPE OPERATOR WITH UNBOUNDED JUMP KERNEL

VICTORIA KNOPOVA

Let

$$Lf(x) = \int_{\mathbb{R}^d} \left( f(x+y) - f(x) - \nabla f(x) u 1_{|u|\le 1} \right) \nu(x,du), \quad f \in C_0^2(\mathbb{R}^d),$$

where $C_0^2(\mathbb{R}^d)$ is the set of twice differentiable functions with compact support, and $\nu(x,du)$ is the Lévy-type kernel, i.e.

$$\int_{\mathbb{R}^d} (|u|^2 \wedge 1)\nu(x,du) < \infty \quad \text{for any } x \in \mathbb{R}^d.$$

If the kernel $\nu(x,du)$ is bounded in $x$ and in some sense regular in $x$, one can show that the Martingale problem for $(L, C_0^2)$ is well posed and its solution is a Feller process which admits a transition probability density. The construction partly relies on the parametrix method, i.e. the transition probability density is constructed in the form

$$p_t(x,y) = p_t^0(x,y) + r_t(x,y),$$

where $p_t^0(x,y)$ is the zero-order approximation and $r_t(x,y)$ is the remainder.

This talk is devoted to the parametrix constriction for the operator $L$ in the case when $\nu(x,du)$ is unbounded. The important assumptions on $\nu(x,du)$ are the regularity of $\nu(x,du)$ in $u$ for $|u| \le 1$, and boundedness of $\nu(x,\{u : |u| \ge 1\})$, which forbids the jumps which are "too big". We discus the choice of the zero-order approximation $p^0$, under which the parametrix construction works.

The talk is based on the on-going work [1] which in in preparation.

## References


[1] V. Knopova, S. Menozzi, in preparation.

[2] V. Knopova, A. Kulik. R. Schilling. Construction and heat kernel estimates of general stable-like Markov processes. *Diss. Math.* **569** (2021), 1–86.

[3] K. Bogdan, V. Knopova, P. Sztonyk. Heat kernel of anisotropic nonlocal operators. *Doc. Math.*, **25** (2020), 1–54.

Wednesday, March 22, 9:00-9:40

# RELLICH INEQUALITIES VIA OPTIMAL HARDY INEQUALITIES ON DISCRETE GRAPHS

YEHUDA PINCHOVER

For a given subcritical discrete Schrödinger operator $H$ on a weighted infinite graph $X$, we first construct a Hardy-weight $w$ which is optimal in the following sense: The operator $H - \lambda w$ is subcritical in $X$ for all $\lambda < 1$, null-critical in $X$ for $\lambda = 1$, and supercritical near any neighborhood of infinity in $X$ for any $\lambda > 1$.

Then we present weighted Rellich-type inequalities with best constants for $H$ on $X$. The corresponding Rellich weights satisfy certain eikonal inequality involving optimal Hardy-weights. Our results rely on a criticality theory for Schrödinger operators on general weighted graphs.

This is a joint work with Matthias Keller and Felix Pogorzelski.

## References


[1] B. Devyver, M. Fraas, and Y. Pinchover, *Optimal Hardy weight for second-order elliptic operator: an answer to a problem of Agmon.* J. Funct. Anal. 266 (2014), 4422–4489.

[2] F. Fischer, *On the optimality and decay of p-Hardy weights on graphs.* arXiv: 2212.07728.

[3] M. Keller, Y. Pinchover, and F. Pogorzelski, *Optimal Hardy inequalities for Schrödinger operators on graphs.* Comm. Math. Phys. 358 (2018), 767–790.

[4] M. Keller, Y. Pinchover, and F. Pogorzelski, *From Hardy to Rellich inequalities on graphs.* Proc. Lond. Math. Soc. (3) 122 (2021), 458–477.

[5] M. Keller, Y. Pinchover, and F. Pogorzelski, *Criticality theory for Schrödinger operators on graphs.* J. Spectr. Theory 10 (2020), 73–114.

Wednesday, March 22, 9:45-10:25

# GAUSSIAN UPPER HEAT KERNEL BOUNDS ON GRAPHS WITH UNBOUNDED GEOMETRY

CHRISTIAN ROSE (JOINT WITH MATTHIAS KELLER)

It is known that the short-time behavior of the heat kernel associated to a Laplacian on a graph differs significantly from Varadhan's well-established short-time behavior of heat kernels on manifolds [2]. Hence, characterizations of Gaussian upper heat kernel bounds on manifolds for small times cannot hold on graphs.

In this talk, which is based on [3] and the ongoing work [4], I will present a characterization of Gaussian upper heat kernel bounds on graphs for large times. The conjunction of such heat kernel bounds together with a modified volume doubling property valid only for large balls is equivalent to Sobolev inequalities valid only on large balls. Distances are thereby encoded via intrinsic metrics adapted to the vertex measure.

Our heat kernel bounds are optimal in the following sense: on one hand, the Gaussian is given by Davies' function which is essentially exact on the integers [1, 5]. On the other hand, it includes the infimum of the spectrum of the Laplacian, resembling Li's long-time asymptotics.

Starting from Sobolev inequalities, the only local geometric information entering the Gaussian upper bounds is the weighted vertex degree, and the corresponding terms become negligible in time. Obtaining Sobolev inequalities from Gaussian upper bounds requires a stronger and more uniform control on the weighted vertex degree.

Note that a characterization of Gaussian upper bounds including all the above features is new even for the normalizing measure.

## References

[1] E. B. Davies. Large Deviations for Heat Kernels on Graphs. *J. Lond. Math. Soc.*, s2-47(1):65–72, 02 1993.

[2] M. Keller, D. Lenz, F. Münch, M. Schmidt, and A. Telcs. Note on short-time behavior of semigroups associated to self-adjoint operators. *Bull. Lond. Math. Soc.*, 48(6):935–944, 2016.

[3] M. Keller and C. Rose. Gaussian upper bounds for heat kernels on graphs with unbounded geometry. Preprint, arXiv:2206.04690 [math.AP], 2022.

[4] M. Keller and C. Rose. Sobolev inequalities and Gaussian upper heat kernel bounds on graphs. In preparation.

[5] M. M. H. Pang. Heat kernels of Graphs. *J. London Math. Soc.*, 47:50–64, 1993.

Wednesday, March 22, 11:00-11:40

# GRADIENT FORMULA FOR TRANSITION SEMIGROUP CORRESPONDING TO STOCHASTIC EQUATION DRIVEN BY A SYSTEM OF INDEPENDENT LÉVY PROCESSES

ENRICO PRIOLA* (JOINT WORK WITH ALEXEI M. KULIK AND SZYMON PESZAT)

Let us consider the non-local operator

$$\mathcal{L}f(x) = \langle b(x), \nabla f(x)\rangle + \sum_{j=1}^{d} \int_{\mathbb{R}} \left( f(x+\xi e_j) - f(x) - 1_{\{|\xi|\le 1\}}\, \xi \frac{\partial f}{\partial x_j}(x) \right) m_j(d\xi)$$

associated to the SDE

$$dX_t^x = b(X_t^x)dt + dZ_t, \qquad X_0^x = x \in \mathbb{R}^d,$$

where $Z = (Z_1, \dots, Z_d)$ is a system of independent real-valued Lévy processes. We discuss the operator $\mathcal{L}$. We concentrate on the relevant case when $m_j(d\xi) = c|\xi|^{-1-\alpha} d\xi$ (the cylindrical $\alpha$-stable case, $\alpha \in (0,2)$). Then, when $b : \mathbb{R}^d \to \mathbb{R}^d$ is regular, using the Malliavin calculus, we establish the following gradient formula

$$\nabla P_t f(x) = \mathbb{E}\left[f\left(X_t^x\right) Y(t,x)\right], \qquad f \in C_b(\mathbb{R}^d), \;\; t > 0,$$

where $(P_t)$ is the associated transition Markov semigroup and the random field $Y$ does not depend on $f$. Sharp $L^1$-estimates for $Y(t,x)$ and uniform estimates on $\nabla P_t f(x)$ are also given.

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Wednesday, March 22, 11:45-12:45

# DISCRETE FEYNMAN–KAC EVOLUTIONS WITH CONFINING POTENTIALS

WOJCIECH CYGAN

We study a certain discrete-time Markov evolution in a countably infinite state space that describes the motion of a single particle which is confined through an unbounded potential. From the probabilistic point of view it is a Markov chain whose paths are killed with random intensity coming from an external potential. Its (non-conservative) transition semigroup is a counterpart of the classical Feynman–Kac semigroup. We are mainly interested in long-range Markov chains whose generators are nonlocal (in a specific sense) discrete operators.

In the talk we will give a short introduction to this topic. We will first discuss sharp estimates for functions that are (sub-)harmonic in infinite sets with respect to the discrete Feynman–Kac operators and some applications to the decay rates of solutions to equations involving nonlocal discrete Schrödinger operators (e.g. fractional powers of the nearest-neighbour Laplacians and related quasi-relativistic operators). These results will be compared with respective estimates for the case of the nearest-neighbour random walk which evolves on an infinite graph of finite geometry. Further, we will investigate the behaviour of the heat kernels of the discrete Feynman-Kac semigroups as wells as the associated ultracontractivity properties.

Our approach is based on the *direct step property* (DSP in short) of the underlying Markov chain and it encompasses a fairly general class of processes and operators. We will present a few constructions leading to DSP Markov chains and illustrate them by various examples.

This is a joint work with Kamil Kaleta (WUST), René Schilling (TU Dresden) and Mateusz Śliwiński (WUST).

Thursday, March 23, 9:00-9:40

# LOW-DIMENSIONAL BESSEL AND CIR PROCESSES

YULIYA MISHURA, ANDREY PILIPENKO, ANTON YURCHENKO-TYTARENKO

We start with Cox-Ingersoll-Ross processes of dimension less than 1. It is well-known that their trajectories are nonnegative with probability 1 but are irregular in the sense that they have a big number of zeros. The same is true for the square root of CIR-process that in particular case when the linear drift is zero, is a Bessel process. We obtain a new equation for the square root of the CIR process utilizing the fact that non-negative diffusion processes can be obtained by the transformation of time and scale of some reflected Brownian motion. As the result, we derive this equation, which contains a term characterized by the local time of the corresponding reflected Brownian motion. Additionally, we establish a new connection between low-dimensional CIR processes and reflected Ornstein-Uhlenbeck processes, providing a new representation of Skorokhod reflection functions. We also prove that the limits of the reflection functions when the dimension tends to 1 from the left and from the right, coincide. More precisely, we consider the process of the form

$$X(t) = x_0 + \int_0^t (a - bX(s))\, ds + \sigma \int_0^t \sqrt{X(s)} dW(s), \tag{1}$$

where $a < \sigma^2/4$, and prove, in particular, the following result.

**Theorem 1.** *Let $X$ be given by (1). Then, with probability 1,*

$$\begin{aligned} L(t) &:= \lim_{\varepsilon \downarrow 0} \frac{1}{2} \int_0^t \left( \frac{a}{\sqrt{X(s)+\varepsilon}} - \frac{\sigma^2}{4} \frac{X(s)}{(X(s)+\varepsilon)^{\frac{3}{2}}} \right) ds \\ &= - \lim_{\varepsilon \downarrow 0} \frac{1}{2} \int_0^t \left( \frac{\frac{\sigma^2}{4} - a}{\sqrt{X(s)+\varepsilon}} - \frac{\sigma^2}{4} \frac{\varepsilon}{(X(s)+\varepsilon)^{\frac{3}{2}}} \right) ds \\ &= -\frac{1}{2} \left( \frac{\sigma^2}{4} - a \right) \int_0^\infty y^{\frac{4a}{\sigma^2} - 2} \left( \ell(t,y) - \ell(t,0) \right) dy, \end{aligned} \tag{2}$$

*where $\ell = \ell(t,y)$ is a local time of a specific time- and space-changed Wiener process, and $L$ is a reflection function of $X^{\frac{1}{2}}$.*

So, we give the new representation of the reflection function, comparing to [1]. Furthermore, we can prove that the limits of reflection functions where $a \to \frac{\sigma^2}{4}$ from the left and from the right coincide (the limit from the right was considered in [2]).

## References


[1] Bertoin, J., Complements on the Hilbert transform and the fractional derivative of Brownian local times. *Kyoto journal of mathematics 30*, 4 (1990), 651–670.

[2] Mishura, Y., and Yurchenko-Tytarenko, A. Standard and fractional reflected Ornstein–Uhlenbeck processes as the limits of square roots of Cox–Ingersoll–Ross processes. *Stochastics. An International Journal of Probability and Stochastic Processes* (2022), 1–19.

Thursday, March 23, 9:45-10:25

# CANONICAL (MARCUS) TRANSPORT EQUATIONS

LENA–SUSANNE HARTMANN AND ILYA PAVLYUKEVICH

We solve a Lévy driven linear stochastic first order partial differential equation (transport equation) understood in the canonical (Marcus) form. The solution is obtained with the help of the method of stochastic characteristics and it has the same form as a solution of a deterministic PDE or a solution of a stochastic PDE driven by a Brownian motion studied by Kunita (1984, 1997).

## References

[1] Hartmann, L.-S. and Pavlyukevich, I. *First order linear Marcus SPDEs*. Stochastics and Dynamics, 2023, to appear, arXiv:2303.00674

Thursday, March 23, 11:00-11:40

## APPROXIMATION RESULTS IN HÖLDER NORMS FOR MILD SOLUTIONS OF STOCHASTIC TIME-FRACTIONAL PDES

MIHÁLY KOVÁCS (AND K. FAHIM AND E. HAUSENBLAS)

We investigate the quality of space approximations of a class of stochastic integral equations of convolution type with Gaussian noise. Such equations arise, for example, when considering mild solutions of stochastic fractional order partial differential equations but also when considering mild solutions of classical stochastic partial differential equations. The key requirement for the equations is a smoothing property of the deterministic evolution operator which is typical in parabolic type problems. We show that if one has access to nonsmooth data estimates for the deterministic error operator together with its derivative of a space discretization procedure, then one obtains error estimates in pathwise Hölder norms with rates that can be read off the deterministic error rates.

### References

[1] K. Fahim, E. Hausenblas and M. Kovács, Some approximation results for mild solutions of stochastic fractional order evolution equations driven by Gaussian noise, *Stoch PDE: Anal Comp*, 2022. https://doi.org/10.1007/s40072-022-00250-0.

Thursday, March 23, 11:45-12:25

## SEMILINEAR EQUATIONS FOR NONLOCAL OPERATORS

VANJA WAGNER (JOINT WORK WITH IVAN BIOČIĆ AND ZORAN VONDRAČEK)

We study semilinear problems in general bounded open sets for non-local operators with exterior and boundary conditions, where the operators are more general than the fractional Laplacian. Let $D \subset \mathbf{R}^d$, $d \geq 2$, be a bounded open set. We consider the existence and uniqueness of solutions to Dirichlet (with complement and boundary data) semilinear problems driven by a certain class of nonlocal operators:

$$\begin{aligned} -Lu(x) &= f(x, u(x)) && \text{in } D \\ u &= \lambda && \text{in } D^c \\ W_D u &= \mu && \text{on } \partial D, \end{aligned}$$

where

- $L$ is a second-order operator of the form $L = -\phi(-\Delta)$, where $\phi : (0, \infty) \to (0, \infty)$ is a complete Bernstein function
- $f : D \times \mathbf{R} \to \mathbf{R}$ is the nonlinearity - e.g. $f(x, t) = |t|^{p-1}t$, $p > 1$
- $\lambda$ is a signed measure on $D^c$
- $W_D$ is a boundary trace operator
- $\mu$ a signed measure on $\partial D$.

We also give results in the case of bounded $C^{1,1}$ open sets. In this setting the conditions are given in terms of the function $V : \langle 0, \infty\rangle \to \langle 0, \infty\rangle$, i.e. the renewal function of the ascending ladder height process of the corresponding 1-dim subordinate Brownian motion.

### References

[1] N. Abatangelo, *Large s-harmonic functions and boundary blow-up solutions for the fractional Laplacian.* Discrete Contin. Dyn. Syst. **35** (2015), 5555–5607.

[2] I. Biočić, *Representation of harmonic functions with respect to subordinate Brownian motion.* JMAA **506** (2022), 125554.

[3] I. Biočić, Z. Vondraček, V. Wagner, *Semilinear equations for non-local operators: beyond the fractional Laplacian.* Nonlinear Analysis, 207 (2021), 112303.

[4] K. Bogdan, S. Jarohs and E. Kania, *Semilinear Dirichlet problem for the fractional Laplacian.* Nonlinear Anal. **193** (2020), 111512.

Thursday, March 23, 12:30-13:10

# HARMONICALLY SMALL SOLUTIONS TO THE DIRICHLET PROBLEM FOR SEMILINEAR PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS

TOMASZ KLIMSIAK

Let $E$ be a locally compact separable metric space, $D$ be an open subset of $E$ and $m$ be a Radon measure on $E$ with full support. Let $(L, \mathfrak{D}(L))$ be a self-adjoint operator that generates a Markov semigroup $(T_t)_{t>0}$ on $L^2(E;m)$ (so-called *Dirichlet operators*). The aim of the talk is to present results on the following semilinear Dirichlet problem:

$$-Lu = f(\cdot,u) + \mu \quad \text{in } D, \qquad u = g \quad \text{on } \partial_\chi D. \tag{1}$$

Here $\partial_\chi D$ is the harmonic boundary of $D$, which in general depends on $L$ (typical examples are $\partial D$ or $D^c$), $f : E \times \mathbb{R} \to \mathbb{R}$, $g : \partial_\chi D :\to \mathbb{R}$ are given functions and $\mu$ is a smooth signed measure (with respect to the capacity $Cap : 2^E \to [0,\infty]$ associated with $L$).

It is now recognized that a well-posed Dirichlet problem for this class of operators must consist of two conditions: exterior/boundary condition (1), and a description of asymptotic behavior of solutions at the boundary $\partial D$. The most general formulation of the last condition known in the literature deals with a subclass of operators of the form

$$Lu(x) = \text{P.V.} \int_{\mathbb{R}^d} (u(y) - u(x)) j(x,y)\, dy, \tag{2}$$

and is based on the notion of *boundary trace operator* $W_D$ introduced in [1]:

$$W_D(u)(A) := \lim_{U \nearrow D} \int_A G_U(x_0, z) \left( \int_{D\setminus U} j(|z-y|)u(y)dy \right) dz, \quad A \subset \mathbb{R}^d, \tag{3}$$

where $G_U$ is the Green function of the operator $L$ restricted to $U$. General form of the semilinear Dirichlet problem within the class of operators (2) admits the form

$$-Lu = f(\cdot,u) + \mu \quad \text{in } D, \qquad u = g \quad \text{on } D^c, \quad W_D(u) = h \quad \text{on } \partial D. \tag{4}$$

Any solution to the above problem, with $h \equiv 0$, is called *harmonically small*.

Our goal is to provide a theory of *harmonically small* solutions to (1) for general Dirichlet operators and arbitrary open sets $D \subset E$. We propose one without using the notion of *boundary trace operator*. One of the main results states the following.

**Theorem 1.** *Assume that $g$ is suitable integrable and*

(A1) *$f$ is a Carathéodory function, $y \mapsto f(x,y)$ is non-increasing for each $x \in E$ (no restrictions on the growth!), and $x \mapsto f(x,y)$ is quasi-integrable for each $y \in E$,*

(A2) *$f(\cdot,0) \in L^1_\rho(D;m)$ and $\int_D \rho\, d|\mu| < \infty$ for some strictly positive $(T^D_t)$-excessive function $\rho$ on $D$ (e.g. $\rho \equiv 1$ or the principal eigenvalue for $L_D$).*

*Then there exists a unique harmonically small solution to (1).*

This is a generalization of an existence result proved in [2] (homogeneous case).

## References


[1] Bogdan, K., Jarohs, S., Kania, E.: Semilinear Dirichlet problems for the fractional Laplacian. *Nonlinear Anal.* **193** (2020), Paper No. 111512, 20 pp.

[2] Klimsiak, T., Rozkosz, A.: Dirichlet forms and semilinear elliptic equations with measure data. *J. Funct. Anal.* **265** (2013) 890–925.

Thursday, March 23, 15:00-15:40

# NON-UNIQUENESS FOR AN INITIAL VALUE PROBLEM FOR EVOLUTION EQUATION WITH FRACTIONAL LAPLACIAN AND REACTION TERM OF SUBLINEAR POWER GROWTH

PETR GIRG (AND JIŘÍ BENEDIKT, VLADIMIR E. BOBKOV, RAJ N. DHARA)

In this talk we will discuss results of our recent work [1] concerning non-uniqueness of finite energy solutions of the problem

$$\begin{cases} u_t + (-\Delta)^s u = q(x)|u|^{\alpha-1}u & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{in } (0,T)\times(\mathbb{R}^N\setminus\Omega), \\ u = 0 & \text{at } t=0\,. \end{cases} \tag{1}$$

Here $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^N$, $N \geq 1$, $T > 0$, $\alpha \in (0,1)$, and $(-\Delta)^s$ stands for the fractional Laplacian corresponding to pointwise definition

$$(-\Delta)^s u(x) = -\frac{s2^{2s}\Gamma(\frac{N+2s}{2})}{\pi^{\frac{N}{2}}\Gamma(1-s)} \lim_{\varepsilon\to 0+} \int_{\mathbb{R}^N\setminus B(x,\varepsilon)} \frac{u(y)-u(x)}{|y-x|^{N+2s}}\,dy\,, \tag{2}$$

for $s \in (0,1)$. We impose the following assumptions on the weight function $q \in L^\infty(\Omega)$: $q \geq 0$ in $\Omega$ and there exists an open subset $O$ of $\Omega$ and a constant $q_0 > 0$ such that $q \geq q_0$ a.e. in $O$.

Let us note that $u \equiv 0$ is the trivial solution of (1). It is shown in [1] that the problem (1) admits nontrivial nonnegative solutions provided $\alpha \in (0,1)$. In this way, we extend non-uniqueness results of the classical work by Fujita and Watanabe [3] for parabolic equations to the fractional Laplacian case.

We work in the class of finite energy solutions and our proof of the non-uniqueness result is based on a construction of suitable super- and sub-solutions ([2, 4]), comparison principles ([5]), and method of monotone iterations.

The nonuniqueness result is complemented by the following uniqueness results: If $\alpha = 1$, then the trivial solution is the unique solution in the class of finite energy solutions of (1). If $\alpha > 1$, then the trivial solution is the unique solution in the class of bounded finite energy solutions of (1).

## References


[1] Benedikt, Jiří; Bobkov, Vladimir E.; Dhara, Raj N., *Nonuniqueness for fractional parabolic equations with sublinear power-type nonlinearity.* Version 1. 13th Feb. 2023. arXiv:2302.06363. https://doi.org/10.48550/arXiv.2302.06363

[2] Dyda, Bartłlomiej, Fractional calculus for power functions and eigenvalues of the fractional Laplacian. *Fractional Calculus and Applied Analysis*, 15(4) (2012), 536–555. http://dx.doi.org/10.2478/s13540-012-0038-8

[3] Fujita, Hiroshi; Watanabe, Shinzo, On the uniqueness and non-uniqueness of solutions of initial value problems for some quasi-linear parabolic equations. *Commun. Pure Appl. Math.* 21 (1968), 631–652. http://dx.doi.org/10.1002/cpa.3160210609

[4] Kwaśnicki, Mateusz, Ten equivalent definitions of the fractional Laplace operator. *Fractional Calculus and Applied Analysis*, 20(1) (2017), 7–51. http://dx.doi.org/10.1515/fca-2017-0002

[5] Leonori, Tommaso; Peral, Ireneo; Primo, Ana; Soria, Fernando, Basic estimates for solutions of a class of non local elliptic and parabolic equations. *Discrete & Continuous Dynamical System*, 35(12) (2015), 6031–6068. http://dx.doi.org/10.3934/dcds.2015.35.6031

Thursday, March 23, 15:45-16:25

# THE MASTER EQUATION FOR THE MEAN FIELD GAME SYSTEMS DRIVEN BY LÉVY OPERATORS IN $\mathbb{R}^d$

ARTUR RUTKOWSKI

We prove existence and uniqueness of solutions to the master equation associated with the mean field game system

$$\begin{cases} -\partial_t u - \mathcal{L}u + H(x,u,Du) = F(x,m(t)) & \text{in } (t_0,T)\times\mathbb{R}^d, \\ \partial_t m - \mathcal{L}^* m - \operatorname{div}(m D_p H(x,u,Du)) = 0 & \text{in } (t_0,T)\times\mathbb{R}^d, \\ m(t_0) = m_0, \qquad u(T,x) = G(x,m(T)), \end{cases} \tag{1}$$

where $\mathcal{L}$ is a Lévy operator, roughly speaking, of order greater than 1. For example, we allow $(-\Delta)^{\alpha/2}$ with $\alpha \in (1,2]$, mixed local-nonlocal operators, and strongly anisotropic stable operators.

In order to obtain our result we establish several well-posedness and regularity results for linear and semilinear equations with Lévy diffusions, which are of their own interest. Furthermore, because of working in the whole space, we need a completely new compactness argument for the linearized system associated with (1).

The master equation can be applied to show that the mean field games are a good approximation of games with a large number of players. This was first done in the seminal work of Cardaliaguet, Delarue, Lasry and Lions [1].

Based on a joint work with Espen R. Jakobsen (NTNU).

## References

[1] P. Cardaliaguet, F. Delarue, J.-M. Lasry, P.-L. Lions. *The master equation and the convergence problem in mean field games*, volume 201 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2019.

Friday, March 24, 9:00-9:40

## LIOUVILLE-TYPE THEOREMS FOR KOLMOGOROV AND ORNSTEIN–UHLENBECK OPERATORS

ALESSIA E. KOGOJ

We collect Liouville-type properties that hold true for Kolmogorov operators with constant coefficients and for their time-stationary counterpart, the Ornstein–Uhlenbeck operators. In particular, we discuss uniqueness results for solutions and sub-solutions in $L^p$-spaces and for solutions in the whole space or in halfspaces bounded just from one-side. Polynomial Liouville properties and a Liouville theorem *at* $t = -\infty$ are also presented.

The results were obtained in collaboration with A. Bonfiglioli, E. Lanconelli, Y. Pinchover, S. Polidoro and E. Priola.

### References

[1] A.E. Kogoj, E. Lanconelli and E. Priola, *Harnack inequality and Liouville-type theorems for Ornstein-Uhlenbeck and Kolmogorov operators,* Math. Eng. 2, **2** (2020), 3491–3494.

[2] A.E. Kogoj, *Liouville-type theorems for Kolmogorov and Ornstein-Uhlenbeck operators,* Lect. Notes Semin. Interdiscip. Mat. 15, **2** (2020), 45–53.

[3] A.E. Kogoj, Y. Pinchover and S. Polidoro, *On Liouville-type theorems and the uniqueness of the positive Cauchy problem for a class of hypoelliptic operators*, J. Evol. Equ. **16** no.4 (2016), 905–943.

[4] A. Bonfiglioli and A.E. Kogoj, *Weighted $L^p$-Liouville Theorems for Hypoelliptic Partial Differential Operators on Lie Groups*, J. Evol. Equ. **16** no.3 (2016), 569–585.

[5] A.E. Kogoj and E. Lanconelli, *$L^p$-Liouville theorems for invariant Partial Differential Equations in* $\mathbb{R}^N$, Nonlinear Anal. **121** (2015), 188–205.

[6] A.E. Kogoj and E. Lanconelli, *Liouville theorems for a class of linear second order operators with nonnegative characteristic form*, Bound. Value Probl. 2007, Art. ID 48232, 16 pp.

Friday, March 24, 9:45-10:25

## VARIATIONS ON LIOUVILLE'S THEOREM

EUGENE SHARGORODSKY (AND DAVID BERGER, RENÉ L. SCHILLING, TEO SHARIA)

A classical theorem of Liouville states that a function that is analytic and bounded on the entire complex plane is in fact constant. The same conclusion is true for a function that is harmonic and bounded on $\mathbb{R}^n$.

The talk is a sequel to [1]. It discusses generalisations of Liouville's theorem to nonlocal translation-invariant operators. It is based on [2] and a further joint work in progress with D. Berger, R.L. Schilling, and T. Sharia. It follows from our results that if $\{\eta \in \mathbb{R}^n \mid m(\eta) = 0\} \subset \{0\}$, then, under suitable conditions, every polynomially bounded solution $f$ of the equation $m(D)f = 0$ is in fact a polynomial (see [2]), while sub-exponentially growing solutions admit analytic continuation to entire functions on $\mathbb{C}^n$.

### References

[1] D. Berger, *Liouville theorems for non-local operators,* MFO-RIMS Tandem Workshop: Nonlocality in Analysis, Probability and Statistics, Mathematisches Forschungsinstitut Oberwolfach, Report No. 15/2022, 17–18 (DOI: 10.4171/OWR/2022/15).

[2] D. Berger, R. L. Schilling, and E. Shargorodsky, *The Liouville theorem for a class of Fourier multipliers and its connection to coupling* (arXiv:2211.08929; submitted).

Friday, March 24, 11:00-11:40

# LIOUVILLE'S THEOREMS FOR LÉVY OPERATORS

MATEUSZ KWAŚNICKI (AND TOMASZ GRZYWNY)

Let $L$ be a Lévy operator (that is, the generator of a Lévy process). A function $h$ is said to be harmonic with respect to $L$ if $Lh = 0$ in an appropriate sense. A Liouville's theorem for an operator $L$ is a result which, under appropriate assumptions, characterises the class of functions harmonic with respect to $L$. The most common variant states that bounded harmonic functions are necessarily constants. For general Lévy operators this was proved recently by Alibaud, del Teso, Endal and Jakobsen in [1], and independently by Berger and Schilling in [3].

Our contribution is three-fold.

(1) First, we prove that *positive* harmonic functions are mixtures of *harmonic exponentials*, in the spirit of Deny's theorem for convolution equations (or random walks; see [6]) and a less general result for Lévy operators given in [3].

(2) Next, we provide a variant of Liouville's theorem for *signed* harmonic functions, which is similar to the result of [4], and which asserts that, under appropriate assumptions, signed harmonic functions are necessarily *harmonic polynomials*. Although not completely general, our result extends all previously known theorems of this kind [1, 2, 4, 5, 7, 8, 10].

(3) Finally, we provide an explicit example of a Lévy operator $L$ and a function $h$ which shows that without any conditions on the Lévy operator $L$ and the harmonic function $h$, Liouville's theorem fails. Namely, we construct a Lévy operator $L$ and a signed, polynomially bounded function $h$ which is harmonic with respect to $L$, but which is not a polynomial.

These results can be found in our recent preprint [9].

## References

[1] N. Alibaud, F. del Teso, J. Endal, E.R. Jakobsen, *The Liouville theorem and linear operators satisfying the maximum principle.* J. Math. Pures Appl. 142(2020): 229–242.

[2] M.T. Barlow, R.F. Bass, C. Gui, *The Liouville property and a conjecture of De Giorgi.* Commun. Pure Appl. Math. 53(8) (2000): 1007–1038.

[3] D. Berger, R.L. Schilling, *On the Liouville and strong Liouville properties for a class of non-local operators.* Math. Scand. 128(2) (2022): 365–388.

[4] D. Berger, R.L. Schilling, E. Shargorodsky, *The Liouville theorem for a class of Fourier multipliers and its connection to coupling.* Preprint, 2022, arXiv:2211.08929.

[5] W. Chen, L. D'Ambrosio, Y. Li, *Some Liouville theorems for the fractional Laplacian.* Nonlinear Anal. 121 (2015): 370–381.

[6] J. Deny, *Sur l'equation de convolution $\mu = \mu * \sigma$.* Séminaire Brelot-Choquet-Deny. Théorie du potentiel, tome 4 (1959–1960), exp. no 5: 1–11.

[7] M.M. Fall, *Entire s-harmonic functions are affine.* Proc. Amer. Math. Soc. 144 (2016): 2587–2592.

[8] M.M. Fall, T. Weth. *Liouville Theorems for a General Class of Nonlocal Operators.* Potential Anal. 45(1) (2016): 187–200.

[9] T. Grzywny, M. Kwaśnicki, *Liouville's theorems for Lévy operators.* Preprint, 2023, arXiv:2301.08540.

[10] F. Kühn, *A Liouville theorem for Lévy generators.* Positivity 25 (2021): 997–1012.

Friday, March 24, 11:45-12:25

## HARDY OPERATORS IN ANGULAR MOMENTUM CHANNELS

KONSTANTIN MERZ (KRZYSZTOF BOGDAN)

For $d \in \mathbb{N}$ and $\alpha \in (0, \min\{2, d\})$, we consider the family of Hardy operators

$$L_\kappa := (-\Delta)^{\alpha/2} - \kappa/|x|^\alpha \quad \text{in } L^2(\mathbb{R}^d)$$

with critical or subcritical coupling constant $\kappa$. These operators frequently arise in mathematical physics, commonly as scaling limits of more complicated problems, ranging from combustion theory to the study of black holes, and to relativistic quantum mechanics.

Over the last years, multipliers $M(L_\kappa)$, such as $M(L_\kappa) = \mathrm{e}^{-tL_\kappa}$, and the corresponding quadratic forms $\langle f, M(L_\kappa) f\rangle_{L^2(\mathbb{R}^d)}$ for general $f \in L^2(\mathbb{R}^d)$ belonging to the form domain of $M(L_\kappa)$, have received considerable attention.

In this talk, we exploit the spherical symmetry, perform an angular momentum decomposition, and consider the quadratic forms

$$\langle [u]_\ell, L_\kappa [u]_\ell\rangle_{L^2(\mathbb{R}^d)} \qquad \text{for } [u]_\ell(x) := u(|x|)Y_\ell(x/|x|) \tag{1}$$

and $L^2(\mathbb{S}^{d-1})$-normalized spherical harmonics $Y_\ell$ separately. Our modest contribution is a first step to understanding the resulting operator and multipliers thereof in more detail. More precisely, we prove a ground state representation for (1), which, in particular, provides an alternative proof of an improved Hardy inequality by Le Yaouanc, Oliver, and Raynal [LYOR97] and Yafaev [Yaf99]. Our proof uses a construction of Bogdan, Dyda, and Kim [BDK16] and requires a detailed analysis of the integral kernel in $\langle [u]_\ell, \exp(-tL_0)[u]_\ell\rangle_{L^2(\mathbb{R}^d)}$.

This is joint work with Krzysztof Bogdan.

### References


[BDK16] Krzysztof Bogdan, Bartłomiej Dyda, and Panki Kim. Hardy inequalities and non-explosion results for semigroups. *Potential Anal.*, 44(2):229–247, 2016.

[LYOR97] A. Le Yaouanc, L. Oliver, and J.-C. Raynal. The Hamiltonian $(p^2 + m^2)^{1/2} - \alpha/r$ near the critical value $\alpha_c = 2/\pi$. *J. Math. Phys.*, 38(8):3997–4012, 1997.

[Yaf99] D. Yafaev. Sharp constants in the Hardy-Rellich inequalities. *Journ. Functional Analysis*, 168(1):212–144, October 1999.

Friday, March 24, 12:30-13:10

# SMALL TIME CHAOS EXPANSIONS OF DIFFUSION HEAT KERNELS

OLEKSII KULYK

The talk is based on joint research with D. Ivanenko (Taras Shevchenko Kyiv Univeristy, Ukraine) and A. Kohatsu-Higa (Ritsumeikan University, Japan), [1], [2] We develop a new type of small time approximations to heat kernels of elliptic diffusions, where the basic (Euler-Maruyama) conditionally Gaussian approximation is improved by adding a proper combination of 'chaos monomials' of the form

$$c^m_{i_1,\dots,i_k}(x)t^m\Phi_t^{(i_1,\dots,i_k)}(x,y), \tag{1}$$

where the coefficients $c^m_{i_1,\dots,i_k}(x)$ depend on the coefficients of the diffusion, and $\Phi_t^{(i_1,\dots,i_k)}(x,y)$ are the higher order Hermite functions corresponding to the zero-order conditionally Gaussian approximation. Such an approximation is unique for any given accuracy rate, which is described in terms of '$t$-order' of the residual kernel. The proof of this main result relies on the classical parametrix representation of the heat kernel combined with an iterated re-arrangement of a finite number of the summands in the parametrix series. Such a re-arrangement combines 'change of the freezing point' in the conditionally Gaussian kernels with Taylor expansion arguments and identities for Hermite functions, and give an algorithm for getting a chaos expansion of arbitrary order.

This algorithm and its computer implementation will be discussed in the talk, together with further applications to statistical inference and simulation for diffusion processes and possible generalization for Lévy-type processes.

## References

[1] S. Bodnarchuk, D. Ivanenko, A. Kohatsu-Higa, A. Kulik, Improved local approximation for multidimensional diffusions: the G-rates, Theory of Probability and Mathematical Statistics. 2019, vol. 101, nr 2, pp. 15-39.

[2] D. Ivanenko, A. Kohatsu-Higa, A. Kulik, Small time chaos approximations for heat kernels of multidimensional diffusions, BIT Numerical Mathematics, https://doi.org/10.1007/s10543-023-00949-z