Poster session, Tuesday, March 21, 15:00-16:25

ESTIMATES OF KERNELS FOR SCHRÖDINGER SEMIGROUPS

MIŁOSZ BARANIEWICZ

We consider the Schrödinger operator of the form $H = -\Delta + V$ acting in $L^2(\mathbb{R}^d, dx)$, $d \geq 1$, where the potential $V : \mathbb{R}^d \to [0, \infty)$ is a locally bounded function. The corresponding Schrödinger semigroup $\left\{e^{-tH} : t \geq 0\right\}$ consists of integral operators, i.e.

$$e^{-tH} f(x) = \int_{\mathbb{R}^d} u_t(x, y) f(y) dy, \quad f \in L^2(\mathbb{R}^d, dx),\ t > 0. \tag{1}$$

I will present new estimates for heat kernel of $u_t(x, y)$. Our results show the contribution of the potential is described separately for each spatial variable, and the interplay between the spatial variables is seen only through the Gaussian kernel.

I will present applications of those theorems for two common classes of potentials. For confining potentials we get two sided estimates and for decaying potentials we get new upper estimate.

The poster is based on joint work with Kamil Kaleta [1].

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Poster session, Tuesday, March 21, 15:00-16:25

THE DOUGLAS FORMULA IN L^p

DAMIAN FAFUŁA (AND KRZYSZTOF BOGDAN, ARTUR RUTKOWSKI)

The classical Douglas formula relates the energy of the harmonic function u on the unit disk $B(0,1) \subset \mathbb{R}^2$ to the energy of its boundary values g on the boundary of the disk, identified with the torus $[0, 2\pi)$:

$$\int\limits_{B(0,1)} |\nabla u(x)|^2 \, dx = \frac{1}{8\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{(g(\eta) - g(\xi))^2}{\sin^2((\xi - \eta)/2)} \, d\eta \, d\xi. \tag{1}$$

The formula is important in the trace theory for Sobolev spaces, since the left-hand side of (1) is the classical Dirichlet integral and the right-hand side is equivalent to the Gagliardo form in $H^{1/2}(\partial B(0,1))$.

We propose an extension of (1):

$$\int\limits_{B(0,1)} |\nabla u(x)|^2 |u(x)|^{p-2} \, dx = \frac{1}{2(p-1)} \int_0^{2\pi} \int_0^{2\pi} \frac{(g(\eta)^{\langle p-1 \rangle} - g(\xi)^{\langle p-1 \rangle})(g(\eta) - g(\xi))}{4\pi \sin^2((\xi - \eta)/2)} \, d\eta \, d\xi.$$

Here and below, we assume that $p \in (1, \infty)$ and $a^{\langle \kappa \rangle} = |a|^\kappa \operatorname{sgn}(a)$ for $a, \kappa \in \mathbb{R}$. In fact, we prove that for all open bounded $C^{1,1}$ sets $D \subseteq \mathbb{R}^d$ with $d \geq 1$, and harmonic functions u in D with boundary values g,

$$\int\limits_D |\nabla u(x)|^2 |u(x)|^{p-2} \, dx = \frac{1}{2(p-1)} \int\limits_{\partial D} \int\limits_{\partial D} (g(z)^{\langle p-1 \rangle} - g(w)^{\langle p-1 \rangle})(g(z) - g(w)) \, \gamma_D(z, w) \, dz \, dw.$$

Here dz, dw refer to the surface measure on ∂D and

$$\gamma_D(z, w) := \partial^z_{\vec{n}} P_D(\cdot, w),$$

is the inward normal derivative of the Poisson kernel P_D of $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ for the set D.

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Poster session, Tuesday, March 21, 15:00-16:25

HARDY–STEIN IDENTITY FOR PURE-JUMP DIRICHLET FORMS

MICHAŁ GUTOWSKI

We want to consider regular conservative pure-jump Dirichlet form:

$$\mathcal{E}(u,v) = \frac{1}{2} \iint\limits_{E\times E\setminus \mathrm{diag}} (u(y)-u(x))(v(y)-v(x))\, J(dx,dy), \quad u,v \in \mathcal{D}(\mathcal{E}),$$

under some mild assumptions. Here J is the *jumping measure*.

Our purpose is to present the Hardy–Stein-type identity

$$\int\limits_E |f|^p\, dm = \int\limits_0^\infty \iint\limits_{E\times E\setminus \mathrm{diag}} F_p(P_t f(x), P_t f(y))\, J(dx,dy)dt \quad f \in L^p(m),$$

where

$$F_p(a,b) := |b|^p - |a|^p - pa^{\langle p-1\rangle}(b-a), \qquad a,b \in \mathbb{R},$$

is the *Bregman divergence* and $a^{\langle\kappa\rangle} := |a|^\kappa \operatorname{sgn}(a)$.

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Poster session, Tuesday, March 21, 15:00-16:25

SEMIFRACTIONAL DERIVATIVES AND SEMISTABLE LÉVY PROCESSES

PETER KERN, SVENJA LAGE, MARK M. MEERSCHAERT

We introduce semi-fractional derivatives using a classical semigroup approach for the class of semistable Lévy processes. These are Lévy processes fulfilling a self-similarity property on a discrete scale and generalize α-stable Lévy processes with continuous scaling property. For $\alpha \in (0,1)$ the semi-fractional derivatives can be seen as generalized fractional derivatives in the sense of Kochubei [6] or as convolution-type derivatives in the sense of Toaldo [7]. Point source solutions of corresponding semi-fractional diffusion equations are given by the probability densities of semistable Lévy processes. We further show that solutions of certain semi-fractional diffusion equations of order $\alpha \in (1,2)$ in space correspond to certain diffusion equations with semi-fractional derivative of order $1/\alpha \in (\frac{1}{2},1)$ in time, called space-time duality in case of ordinary fractional derivatives [1, 2]. In addition, we develop a Grünwald-Letnikov type formula to compute semi-fractional derivatives of order $\alpha \in (0,2) \setminus \{1\}$ numerically. The presentation is based on results in [3, 4, 5].

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Poster session, Tuesday, March 21, 15:00-16:25

SHARP FRACTIONAL HARDY INEQUALITIES WITH A REMAINDER FOR $1 < p < 2$

MICHAŁ KIJACZKO

Our purpose is to present (weighted) fractional Hardy inequalities with a remainder and fractional Hardy–Sobolev–Maz'ya inequalities valid for $1 < p < 2$. We provide a general nonlinear ground-state representation, being a generalisation of the well-known result of Frank and Seiringer [1] to the case $1 < p < 2$.

References

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Poster session, Tuesday, March 21, 15:00-16:25

MEASURABLE SEMIFLOWS GENERATED BY DIFFERENTIAL EQUATIONS

MAREK KRYSPIN

In our research, we focused on measurable semiflows generated by systems of ordinary differential equations with delay and parabolic differential equations with changing delay. Such systems are important for mathematical ecology to study dependence between species. Our results concern dynamical behavior like Lyapunov exponents, Floquet subspaces, exponential separation in such semiflows and its continuous dependence on coefficients (see [1, 2, 3, 4, 5] for more details).

Based on a joint work with Janusz Mierczyński, Sylvia Novo and Rafael Obaya.

References

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Poster session, Tuesday, March 21, 15:00-16:25

HEAT CONTENT FOR LÉVY PROCESSES

JULIA LENCZEWSKA

Let $d \in \mathbb{N}$ and $\mathbf{X} = (X_t)_{t\geq 0}$ be a Lévy process in $\mathbb{R}^d$. For an open set $\Omega \subset \mathbb{R}^d$, we consider the quantity

$$H_\Omega(t) = \int_\Omega \mathbb{P}^x(X_t \in \Omega)\,\mathrm{d}x,$$

called the heat content related to the process $\mathbf{X}$. It is known [1] that if $\mathbf{X}$ has finite variation, then

$$H_\Omega(t) = |\Omega| - t\,\mathrm{Per}_{\mathbf{X}}(\Omega) + o(t) \qquad \text{as} \quad t \to 0^+,$$

where $\mathrm{Per}_{\mathbf{X}}$ is the perimeter related to the process $\mathbf{X}$. We establish the next terms of the asymptotic expansion of $H_\Omega(t)$, under mild assumptions on the characteristic exponent ψ of the process $\mathbf{X}$. Our results are new even for the α-stable processes.

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Poster session, Tuesday, March 21, 15:00-16:25

HEAT KERNELS OF DISCRETE FEYNMAN–KAC OPERATORS

MATEUSZ ŚLIWIŃSKI (JOINT WORK WITH WOJCIECH CYGAN, KAMIL KALETA AND RÉNE SCHILLING)

We present results of our investigation of a particular discrete-time counterpart of the Feynman–Kac semigroup with a confining potential in a countably infinite space. These findings are a continuation of our work, described in detail in the paper [1]. We focus on Markov chains with the direct step property, which is satisfied by a wide range of typically considered kernels. We propose a characterization for asymptotic intrinsic ultracontractivity (aIUC), expressed in terms of transition densities of the underlying Markov chains as well as potentials appearing in the semigroups. We also investigate links between aIUC of Feynman–Kac semigroups and the uniform ergodicity of their intrinsic semigroups.

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Poster session, Tuesday, March 21, 15:00-16:25

ROBUST NONLINEAR NONLOCAL TRACE SPACES

FLORIAN GRUBE (JOINT WORK WITH MORITZ KASSMANN)

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. We answer the following question: For which functions $g : \Omega^c \to \mathbb{R}$ can we find a weak solution to the Dirichlet problem

$$\begin{cases} (-\Delta)_p^s u = 0 & \text{in } \Omega \\ u = g & \text{on } \Omega^c. \end{cases}$$

Here $(-\Delta)_p^s$ is the fractional p-Laplacian for $s \in (0,1)$ and $1 < p < \infty$ defined by

$$(-\Delta)_p^s u(x) := (1-s) \lim_{\varepsilon \to 0^+} \int_{B_\varepsilon(x)^c} |u(x)-u(y)|^{p-2} \frac{u(x)-u(y)}{|x-y|^{d+sp}} dy.$$

By a nonlocal Green-Gauß formula, the appropriate function space for weak solutions is $V^{s,p}(\Omega \,|\, \mathbb{R}^d) := \{u : \mathbb{R}^d \to \mathbb{R} \text{ m.b.} \,|\, [u]_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)} < \infty\}$, where

$$[u]_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)} := (1-s) \int_{(\Omega^c \times \Omega^c)^c} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}} \mathrm{d}(x,y),$$

endowed with the norm $\|u\|_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)} = \left(\|u\|_{L^p(\Omega)}^p + [u]_{V^{s,p}(\Omega \,|\, \mathbb{R}^d)}\right)^{1/p}$. We are particularly interested in the asymptotics as $s \to 1^-$. Recall that $-(-\Delta)_p^s$ converges to the p-Laplacian up to a constant.

We provide Banach spaces $\mathcal{T}^{s,p}(\Omega^c)$ of functions defined on Ω^c such that trace map

$$\mathrm{Tr}_s : V^{s,p}(\Omega \,|\, \mathbb{R}^d) \to \mathcal{T}^{s,p}(\Omega^c), \qquad u \mapsto u|_{\Omega^c}$$

is continuous and linear and there exists a continuous linear right inverse

$$\mathrm{Ext}_s : \mathcal{T}^{s,p}(\Omega^c) \to V^{s,p}(\Omega \,|\, \mathbb{R}^d).$$

The norm of the operators depend on Ω, a lower bound on s as well as a lower and upper bound on p. Furthermore, we recover in the limit:

$$\|\mathrm{Tr}_s u\|_{\mathcal{T}^{s,p}(\Omega^c)} \to \|u|_{\partial\Omega}\|_{W^{1-1/p,p}(\partial\Omega)} \text{ as } s \to 1^-$$

for any $u \in W^{1,p}(\mathbb{R}^d)$.

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